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LINEAR-TOPOLOGICAL CLASSIFICATION OF SEPARABLE L_p-SPACES ASSOCIATED WITH VON NEUMANN ALGEBRAS OF TYPE I

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ABSTRACT

We classify, up to a linear-topological isomorphism, all separable L_p-spaces, $1 \le p < \infty$, associated with von Neumann algebras of type I. In particular, any L_p-space associated with an infinite-dimensional atomic von Neumann algebra is isomorphic to l_p, or to C_p, or to $S_p = (\sum_{n=1}^{\infty} C_p^n)_{l_p}$. Further, any L_p-space, $p \in [1,\infty)$, $p \neq 2$ associated with an infinite-dimensional von Neumann algebra $\mathcal{M}$ of type I is isomorphic to one of the following nine Banach spaces:

$$l_p,\ L_p,\ S_p,\ C_p,\ S_p \oplus L_p,\ L_p(S_p),\ C_p \oplus L_p,\ L_p(C_p),\ C_p \oplus L_p(S_p).$$

In the case $p = 1$ all the spaces in this list are pairwise non-isomorphic.

0. Introduction

Let $\mathcal{M}_i$ be a semifinite von Neumann algebra, let τ_i be a normal faithful semifinite trace on $\mathcal{M}_i$, let $\widetilde{\mathcal{M}_i}$ be a $*$-algebra of all τ_i-measurable operators affiliated with $\mathcal{M}_i$, $1 = 1,2$ (see [FK]). $L_p(\mathcal{M}_i,\tau_i)$, $1 \le p < \infty$, is the Banach space of all operators $A \in \widetilde{\mathcal{M}_i}$ such that $\tau_i(|A|^p) < \infty$ with the norm $\|A\|_p := (\tau_i(|A|^p))^{1/p}$, where $|A| = (A^*A)^{1/2}$, $i = 1,2$. The description of isometric maps between L_p-spaces $L_p(\mathcal{M}_1,\tau_1)$ and $L_p(\mathcal{M}_2,\tau_2)$, $p \in [1,\infty)$, $p \neq 2$, is well-known (see [Y]) and using this description it is easy to see (Corollary 1.5 below) that the latter two spaces are linearly isometric, if and only if the von Neumann algebras $\mathcal{M}_1$ and $\mathcal{M}_2$

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are Jordan $*$-isomorphic. So, the isometric classification of non-commutative L_p-spaces coincides with the classification of von Neumann algebras, up to a Jordan $*$-isomorphism. In particular, there exist uncountably many non-isometric L_p-spaces associated with von Neumann algebras of type I.

A completely different situation appears if we replace an isometry between $L_p(\mathcal{M}_1, \tau_1)$ and $L_p(\mathcal{M}_2, \tau_2)$ by an isomorphism (= continuous linear-topological bijection). In this case, it is possible that $L_p(\mathcal{M}_1, \tau_1)$ and $L_p(\mathcal{M}_2, \tau_2)$, $p \in [1, \infty)$, are isomorphic, although there is no Jordan $*$-isomorphism between $\mathcal{M}_1$ and $\mathcal{M}_2$. The following question arises naturally.

What is the linear-topological classification of non-commutative L_p-spaces?

Our motivation in considering this problem comes mainly from the following three sources: from the famous book of Banach [B] where it is established that the spaces l_p and L_p are not isomorphic (unless $p = 2$), from Ch. McCarthy's result [M] that there is no isomorphic embedding of C_p into L_p (see also [GL]) and from the paper of J. Arazy and J. Lindenstrauss [AL] who showed that there is no isomorphic embedding of L_p into C_p. McCarthy's result was extended in [Su2] by showing that there is no isomorphic embedding of C_p, $2 < p < \infty$, into any L_p-space associated with a finite von Neumann algebra.

Our main result in the present paper concerns the given question in the setting of separable L_p-spaces and von Neumann algebra of type I.

THEOREM 0.1: *Let $\mathcal{M}$ be an infinite-dimensional von Neumann algebra of type I acting in a separable Hilbert space H, let τ be a normal faithful semifinite trace on $\mathcal{M}$, let $L_p(\mathcal{M}, \tau)$, $p \in [1, \infty)$, $p \neq 2$, be the L_p-space associated with $\mathcal{M}$. Then*

(a) *the space $L_p(\mathcal{M}, \tau)$ is isomorphic to one of the following nine spaces:*

$$(\mathbf{L}) \quad l_p,\ L_p,\ S_p,\ C_p,\ S_p \oplus L_p,\ L_p(S_p),\ C_p \oplus L_p,\ L_p(C_p),\ C_p \oplus L_p(S_p);$$

(b) *if (E, F) is a pair of distinct spaces from $(\mathbf{L})$, which does not coincide with the pair $(L_p(C_p),\ C_p \oplus L_p(S_p))$, then E is not isomorphic to F;*

(c) *all nine spaces from $(\mathbf{L})$ are pairwise non-isomorphic, provided $p = 1$.*

Remark 0.2: If (E, F) coincides with the pair $(L_p(C_p),\ C_p \oplus L_p(S_p))$, then it is easy to see that F is isomorphic to a complemented subspace of E. The converse seems to be false and we conjecture that the spaces $L_p(C_p)$ and $C_p \oplus L_p(S_p)$ are non-isomorphic as well, but at the moment we can confirm this hypothesis only in the special case $p = 1$.

In the first section we make some preliminary observations concerning non-commutative L_p-spaces, in particular it is proved that $L_p(\mathcal{M}_1, \tau_1)$ and $L_p(\mathcal{M}_2, \tau_2)$, $p \neq 2$, are isometric if and only if there exists a Jordan $*$-isomorphism between $\mathcal{M}_1$ and $\mathcal{M}_2$. In the second section, we study the nine spaces listed in the assertion of Theorem 0.1, and prove parts (a) and (b) of this theorem. In the third section (via recent results concerning the Dunford–Pettis property in the preduals to von Neumann algebras) we strengthen part (b) by showing that in the case $p = 1$, the space $L_1(C_1)$ is not isomorphic to $C_1 \oplus L_1(S_1)$, and this establishes part (c) of Theorem 0.1. Thus, part (c) of Theorem 0.1 yields a complete linear-topological classification of the preduals to von Neumann algebras of type I.

Our notation and terminology are standard. We refer to [LT 1,2] for Banach space theory, to [BR], [Sa], [SZ], [T] for von Neumann algebras theory and to [FK], [Se] for non-commutative integration theory.

Some results of the present paper were announced in [SC1,2]. The author thanks J. Arazy and V. Chilin for constructive discussions and P. Dodds for his interest.

1. Preliminaries

Recall that the dual of $L_p(\mathcal{M}, \tau)$, $1 < p < \infty$, can be identified with $L_q(\mathcal{M}, \tau)$, $p^{-1} + q^{-1} = 1$. An element g of $L_q(\mathcal{M}, \tau)$ determines a linear functional $\langle \cdot, g \rangle$ on $L_p(\mathcal{M}, \tau)$ by the formula

$$\langle \cdot, g \rangle = \tau(\cdot, g).$$

Therefore $L_p(\mathcal{M}, \tau)$ is reflexive for $1 < p < \infty$. The (infinite-dimensional) space $L_1(\mathcal{M}, \tau)$ is not reflexive, the dual of $L_1(\mathcal{M}, \tau)$ is $\mathcal{M}$. Every element g of $\mathcal{M}$ determines a linear functional on $L_1(\mathcal{M}, \tau)$ by the same formula as above. If $\mathcal{M} = L_\infty(0,1)$ (respectively $l_\infty = l_\infty(\mathbb{N})$) and trace τ is the integral with respect to Lebesgue measure m on the interval $(0,1)$ (respectively, with respect to counting measure on $\mathbb{N}$) then $L_p(\mathcal{M}, \tau)$ coincides with $L_p = L_p(0,1)$ (respectively, l_p). If $\mathcal{M} = B(l_2)$, i.e. $\mathcal{M}$ is the algebra of all bounded linear operators on l_2 and $\tau = tr$ is the standard trace on $B(l_2)$, then $L_p(\mathcal{M}, \tau)$ coincides with the Schatten–von Neumann p-class C_p of compact operators on Hilbert space l_2.

Let $(e_n)_{n=1}^\infty$ be a standard unit vector basis of l_2 and let C_p^n denote the space of all operators A on the n-dimensional Hilbert space $l_2^n = [e_k]_{k=1}^n$ with the norm $\|A\|_p = (tr(x^*x)^{p/2})^{1/p}$, in other words $C_p^n = L_p(B(l_2^n), \mathrm{tr})$. It is clear that the space

$$S_p = (C_p^1 \oplus C_p^2 \oplus \cdots \oplus C_p^n \oplus \cdots)_p$$

can be identified with $L_p(\mathcal{N},\tau)$ where

$$\mathcal{N} = \bigoplus_{n=1}^{\infty} B(l_2^n)$$

is the direct sum of von Neumann algebras $B(l_2^n)$, $n = 1, 2, \ldots$.

Below we consider infinite-dimensional L_p-spaces only and, if it is not specified directly, always assume that the index p belongs to $[1,\infty)$ and $p \neq 2$.

PROPOSITION 1.1: *If $L_p(\mathcal{M},\tau)$, $1 \leq p < \infty$, is a separable Banach space, then there exists a separable Hilbert space H such that $\mathcal{M}$ is $*$-isomorphic to a von Neumann subalgebra of $B(H)$.*

Proof: Recall the definition of the measure topology in $\widetilde{\mathcal{M}}$ generated by the trace τ. This topology is defined by the fundamental system of neighbourhoods around zero $\{U(\epsilon,\delta) : \epsilon, \delta > 0\}$ (see [FK]) where

$$U(\epsilon,\delta) = \{x \in \widetilde{\mathcal{M}} : \|xp\|_{\mathcal{M}} \leq \epsilon,\ \tau(\mathbf{1}-p)) \leq \delta \text{ for some } p \in \mathcal{P}_{\mathcal{M}}\};$$

where $\|\cdot\|_{\mathcal{M}}$ is the C^*-norm on $\mathcal{M}$, $\mathcal{P}_{\mathcal{M}}$ is a complete lattice of all projections from $\mathcal{M}$ and $\mathbf{1}$ is the unit of $\mathcal{M}$. It follows from [Su1], [Me] that if $L_p(\mathcal{M},\tau)$ is a separable Banach space then $\widetilde{\mathcal{M}}$ is separable in the topology τ, and therefore (also by [Su1], [Me]) we conclude that $H = L_2(\mathcal{M},\tau)$ is a separable Hilbert space. So, by [BR] Theorem 2.7.14, there is a normal $*$-isomorphism π of $\mathcal{M}$ into $B(H)$ such that $\pi(\mathcal{M}) = \pi(\mathcal{M})''$. ∎

PROPOSITION 1.2: *Let $\mathcal{M}$ be a semifinite von Neumann algebra acting in a separable Hilbert space H. Then $L_p(\mathcal{M},\tau)$ is a separable Banach space.*

Proof: It follows from [Sa] Proposition 2.1.10 that $L_1(\mathcal{M},\tau)$ is separable and the same arguments as in the proof of Proposition 1.1 complete the proof. ∎

In the sequel, we shall always assume that the von Neumann algebra $\mathcal{M}$ acts in a separable Hilbert space H. It follows that $\mathcal{M}$ is σ-finite (see [BR], [SZ] p.84, [Sa] p.80), in other words any family of mutually orthogonal projections is at most countable.

Further, if

$$U\colon L_p(\mathcal{M}_1,\tau_1) \longrightarrow L_p(\mathcal{M}_2,\tau_2)$$

is a surjective isometry, then (see [Y]) there exists a unitary operator $W \in \mathcal{M}_2$, a positive (possibly, unbounded) operator B affiliated with the center of $\mathcal{M}_2$ and a Jordan $*$-isomorphism $J\colon \mathcal{M}_1 \mapsto \mathcal{M}_2$ such that

$$U(T) = WBJ(T), \quad T \in L_p(\mathcal{M}_1,\tau_1) \cap \mathcal{M}_1$$

and such that

$$\tau_2(B^p J(T)) = \tau_1(T)$$

for every positive $T \in \mathcal{M}_1$. Suppose now that there exists a Jordan $*$-isomorphism between $\mathcal{M}_1$ and $\mathcal{M}_2$ (not necessarily trace preserving). Does it follow that $L_p(\mathcal{M}_1, \tau_1)$ is isometric to $L_p(\mathcal{M}_2, \tau_2)$ for all $1 \leq p < \infty$? The affirmative answer is given below.

PROPOSITION 1.3: *Let τ and ν be semifinite normal faithful traces on a semifinite von Neumann algebra $\mathcal{M}$. Then $L_p(\mathcal{M}, \tau)$ and $L_p(\mathcal{M}, \nu)$ are isometric.*

Proof: By [Se], there exists a positive operator S affiliated with the center of $\mathcal{M}$ such that $\nu(T) = \tau(ST)$ for every positive $T \in \mathcal{M}$. We shall show that the map U from $L_p(\mathcal{M}, \nu)$ into $\widetilde{\mathcal{M}}$ defined by

$$U(T) = S^{1/p}T$$

is a surjective isometry from $L_p(\mathcal{M}, \nu)$ onto $L_p(\mathcal{M}, \tau)$. Indeed, since

$$|S^{1/p}T|^p = |S^{1/p}|^p|T|^p = S|T|^p$$

we have

$$\tau(|U(T)|^p) = \tau(|S^{1/p}T|^p) = \tau(S|T|^p) = \nu(|T|^p).$$

The latter means that $U(T) \in L_p(\mathcal{M}, \tau)$ and $\|U(T)\|_{L_p(\mathcal{M},\tau)} = \|T\|_{L_p(\mathcal{M},\nu)}$. Further, let $z(S)$ be the central support of the operator S. If $z(S) \neq \mathbf{1}$, then there exists a projection $Q \in \mathcal{M}$ such that $Qz(S) = 0, 0 < \nu(Q) < \infty$. We have then $\nu(Q) = \tau(SQ) = \tau(Sz(S)Q) = 0$. This contradiction shows that $z(S) = \mathbf{1}$. Hence there is a positive operator S^{-1} affiliated with the center of $\mathcal{M}$. Let A be an arbitrary element from $L_p(\mathcal{M}, \tau)$ and let

$$T' = (S^{-1})^{1/p}A.$$

We have then

$$\nu(|T'|^p) = \tau(SS^{-1}|A|^p) = \|A\|^p_{L_p(\mathcal{M},\tau)}.$$

It follows that $T' \in L_p(\mathcal{M}, \nu)$ and, since $U(T') = A$, it further follows that U is a bijection. ∎

PROPOSITION 1.4: *Let $(\mathcal{M}_i, \tau_i)$ be two semifinite von Neumann algebras equipped with faithful normal traces τ_i and let J: $\mathcal{M}_1 \to \mathcal{M}_2$ be a Jordan*

$*$-isomorphism between them. Then $L_p(\mathcal{M}_1,\tau_1)$ and $L_p(\mathcal{M}_2,\tau_2)$ are isometric for all $1 \le p < \infty$.

Proof: Since J is a completely additive map such that $J(T) \ge 0$ if and only if $T \ge 0$, it follows that the form ν defined by setting

$$\nu(T) = \tau_2(J(T)) \qquad T \in \mathcal{M}_1$$

is a faithful weight on $\mathcal{M}_1$. Further, for an unitary operator $u \in \mathcal{M}_1$ the operator $J(u)$ is a unitary operator from $\mathcal{M}_2$ ([BR] pp. 217–218) and therefore for any projection $P \in \mathcal{M}_1$

$$\nu(u^*Pu) = \tau_2(J(u^*Pu)) = \tau_2(J(u)^*J(P)J(u)) = \tau_2(J(P)) = \nu(P).$$

In other words ν is a normal faithful semifinite trace on $\mathcal{M}_1$. By Proposition 1.3 there exists an isometry U from $L_p(\mathcal{M}_1,\tau_1)$ onto $L_p(\mathcal{M}_1,\nu)$. Consider the map W from the set $U^{-1}\big(L_p(\mathcal{M}_1,\nu)\cap\mathcal{M}_1\big) \subseteq L_p(\mathcal{M}_1,\tau_1)$ into $\widetilde{\mathcal{M}_2}$ defined by

$$W(T) = JU(T), \quad T \in U^{-1}(L_p(\mathcal{M}_1,\tau_1)\cap\mathcal{M}_1).$$

It is clear that W is an injective linear map. Since $J(A^n) = J(A)^n$ for every $0 \le A \in \mathcal{M}_1$ and all $n = 1,2,\dots$ we have

$$\|J(A^p) - J(P_n(A))\|_{\mathcal{M}_2} = \|J(A^p) - P_n(J(A))\|_{\mathcal{M}_2}$$

where $P_n(t) = \sum_{i=1}^n a_i t^n$ is such that $\|t^p - P_n(t)\|_{L_\infty(0,\|A\|_{\mathcal{M}_1})} \to 0$. Since

$$[0,\|A\|_{\mathcal{M}_1}] = [0,\|J(A)\|_{\mathcal{M}_2}]$$

we get

$$\|J(A)^p - P_n(J(A))\|_{\mathcal{M}_2} \to 0.$$

It follows that

$$J(A)^p = J(A^p)$$

for every $0 \le A \in \mathcal{M}_1, p \in [1,\infty)$. Thus, for any $T \in L_p(\mathcal{M}_1,\tau_1)\cap\mathcal{M}_1$ we have

$$\begin{aligned}\|W(T)\|^p_{L_p(\mathcal{M}_2,\tau_2)} &= \tau_2(|W(T)|^p) = \tau_2(|JU(T)|^p) = \tau_2((J(|U(T)|))^p)\\ &= \tau_2(J(|U(T)|^p)) = \nu(|U(T)|^p) = \|T\|^p_{L_p(\mathcal{M}_1,\tau_1)}.\end{aligned}$$

It follows that W sends $U^{-1}(L_p(\mathcal{M}_1,\tau_1)\cap\mathcal{M}_1)$ into $L_p(\mathcal{M}_2,\tau_1)\cap\mathcal{M}_2$ and it is easy to see that in fact it is a surjective isometry between those two spaces. Since $L_p(\mathcal{M}_i,\tau_i)\cap\mathcal{M}_i$ is dense in $L_p(\mathcal{M}_i,\tau_i)$, $i = 1,2$ [CS] and since U^{-1} is an isometry we infer that W may be extended to a surjective isometry between $L_p(\mathcal{M}_1,\tau_1)$ and $L_p(\mathcal{M}_2,\tau_2)$. ∎

COROLLARY 1.5: *$L_p(\mathcal{M}_1,\tau_1)$ and $L_p(\mathcal{M}_2,\tau_2)$ are isometric for some $p\in[1,\infty)$, $p\neq 2$, if and only if there exists a Jordan $*$-isomorphism between $\mathcal{M}_1$ and $\mathcal{M}_2$.*

For Banach spaces X,Y we use the notation $X\approx Y$, $X\hookrightarrow Y$ and $X\overset{c}{\hookrightarrow}Y$ to denote that X is isomorphic to Y, to a subspace of Y, or to a complemented subspace of Y, respectively. If $X\approx Y$, then

$$d(X,Y):=\inf\{\|T\|\cdot\|T^{-1}\|: T \text{ is an isomorphism from } X \text{ onto } Y\}<\infty.$$

We denote by $l_p(X)$, $1\le p<\infty$, the space of all sequences $x=(x_1,x_2,\dots)$ with $x_j\in X$ and $\sum_{j=1}^{\infty}\|x_j\|_X^p<\infty$ normed by

$$\|x\|_{l_p(X)}:=\left(\sum_{j=1}^{\infty}\|x_j\|_X^p\right)^{1/p}.$$

It is easy to see that $l_p(X)\approx l_p(l_p(X))$ for every Banach space X and every $p\in[1,\infty)$. For the element $f(\cdot)x$ from the Bochner–Lebesgue space $L_p(X)=L_p([0,1],X)$ we shall employ the notation $f\otimes x$. For all $p\in[1,\infty)$, we shall freely identify the space $L_p([0,1],\ L_p(\mathcal{M},\tau))$ with the L_p-space associated with the von Neumann algebra $(L_\infty(0,1)\bar{\otimes}\mathcal{M}, m\otimes\tau)$ (see [BGM], Lemma 6.2).

The following proposition is proved by a simple application of the decomposition method (see [LT 1], [Mi]).

PROPOSITION 1.6: *Let X,Y be Banach spaces such that $X\overset{c}{\hookrightarrow}Y$, $Y\overset{c}{\hookrightarrow}X$ and $l_p(X)\approx X$ for some $1\le p<\infty$. Then $X\approx Y$.*

The proof of the following corollary may be easily obtained from Proposition 1.6 and is therefore omitted.

COROLLARY 1.7: *If X is one of the spaces listed in Theorem 0.1 (a), then $X\approx l_p(X)$.*

2. Parts (a) and (b) of Theorem 0.1

The following two propositions deal with the simplest subclasses of non-commutative L_p-spaces.

PROPOSITION 2.1: *If $\mathcal{M}$ is a commutative von Neumann algebra and τ is a normal faithful semifinite trace on $\mathcal{M}$, then $L_p(\mathcal{M},\tau)\approx L_p(0,1)$, or $L_p(\mathcal{M},\tau)\approx l_p$.*

Proof: Since $\mathcal{M}$ is a finite, σ-finite von Neumann algebra, it follows that there exists a finite faithful trace ν on $\mathcal{M}$ (see [SZ] E.7.4). Therefore, via Proposition

1.3, it suffices to prove the assertion of Proposition 2.1 for $L_p(\mathcal{M},\nu)$. Let

$$p=\bigvee\{q\colon q\in\mathcal{P}_{\mathcal{M}},\ q \text{ is an atom in } \mathcal{P}_{\mathcal{M}}\}.$$

If $p = \mathbf{1}$, then von Neumann algebra $\mathcal{M}$ is $*$-isomorphic to l_∞, whence, via Proposition 1.4,

$$L_p(\mathcal{M},\tau)\approx l_p.$$

If $p\neq\mathbf{1}$, then $\mathcal{M}=(\mathbf{1}-p)\mathcal{M}\oplus p\mathcal{M}$. It is easy to see that $p\mathcal{M}$ is $*$-isomorphic to $L_\infty(0,1)$ (see, for example, [CS] Lemma 4.1), whence, via Proposition 1.6, we have

$$L_p(\mathcal{M},\tau)\approx L_p. \qquad \blacksquare$$

Recall that $\mathcal{M}$ is called an atomic von Neumann algebra if and only if

$$\bigvee\{q\colon q\in\mathcal{P}_{\mathcal{M}},\ q \text{ is an atom in } \mathcal{P}_{\mathcal{M}}\}=\mathbf{1}.$$

PROPOSITION 2.2: *If $\mathcal{M}$ is an atomic von Neumann algebra and τ is a normal faithful semifinite trace on $\mathcal{M}$, then $L_p(\mathcal{M},\tau)\approx l_p$, or $L_p(\mathcal{M},\tau)\approx S_p$, or $L_p(\mathcal{M},\tau)\approx C_p$.*

Proof: Let $Z(\mathcal{M})$ be the center of von Neumann algebra $\mathcal{M}$ and let

$$p_1:=\bigvee\{p \text{ is an atom in } \mathcal{P}_{Z(\mathcal{M})}\colon p\mathcal{M} \text{ is of type } I_n,\ \text{for some } n\in\mathbb{N}\}$$

and

$$p_2:=\bigvee\{p \text{ is an atom in } \mathcal{P}_{Z(\mathcal{M})}\colon p\mathcal{M} \text{ is of type } I_\infty\}.$$

It is clear that p_1 and p_2 are mutually orthogonal central projections such that $p_1+p_2=\mathbf{1}$. Further, it is clear that

$$\mathcal{M}=p_1\mathcal{M}\oplus p_2\mathcal{M}$$

where $p_1\mathcal{M}$ is of type I_{fin} and $p_2\mathcal{M}$ is of type I_∞. If $p_2\neq 0$, then $p_2\mathcal{M}=\bigoplus q_n\mathcal{M}$ where $q_n\in\mathcal{P}_{Z(\mathcal{M})}$ and $q_n\mathcal{M}$ is a factor of type I_∞. It easily follows (via Proposition 1.6) that $L_p(p_2\mathcal{M},\tau)\approx C_p$ and, since $L_p(p_1\mathcal{M},\tau)\overset{c}{\hookrightarrow}C_p$, we get

$$L_p(\mathcal{M},\tau)\approx C_p.$$

If $p_2=0$, then, depending on the fact whether the supremum of the indices n from the definition of p_1 is finite or infinite, we have respectively $L_p(\mathcal{M},\tau)\approx l_p$ or $L_p(\mathcal{M},\tau)\approx S_p$. $\blacksquare$

Remark 2.3: Recall that there is no isomorphic embedding of L_p into l_p (see [B]), of S_p into L_p (see [M], [GL], [P]), or of C_p into S_p (see [AL]).

Let now $\mathcal{M}$ be a purely non-atomic von Neumann algebra, i.e. we shall assume that $z\mathcal{M}$ is atomic for some projection $z \in \mathcal{P}_{Z(\mathcal{M})}$ if and only if $z = 0$. In this case, using [T] Theorem V.1.31 we may assert that $\mathcal{M}$ is Jordan $*$-isomorphic to a (countable) direct sum of von Neumann algebras

$$\bigoplus_{\alpha} L_\infty(0,1) \otimes B(H_{n_\alpha}),$$

where $B(H_{n_\alpha})$ is the algebra of all bounded linear operators over n-dimensional Hilbert space H_{n_α}, $1 \le n_\alpha \le \infty$. It now follows from Proposition 1.4 that $L_p(\mathcal{M},\tau)$ is isomorphic to the l_p-sum of Banach spaces $L_p(C_p^n)$. It is easy to see (via Propositions 1.6, 1.7) that either $L_p(\mathcal{M},\tau)$ is isomorphic to L_p (in the case when the supremum of indices n in the previous l_p-sum is finite), or to $L_p(S_p)$ (when this supremum is infinite, but there are no infinite values of n), or to $L_p(C_p)$ (when there exists at least one infinite value of n). Thus, we have established the following result.

Proposition 2.4: *If $\mathcal{M}$ is a purely non-atomic von Neumann algebra of type I and τ is a normal faithful semifinite trace on $\mathcal{M}$, then $L_p(\mathcal{M},\tau) \approx L_p$, or $L_p(\mathcal{M},\tau) \approx L_p(S_p)$, or $L_p(\mathcal{M},\tau) \approx L_p(C_p)$.*

Noting that any von Neumann algebra of type I may be written as a direct sum of atomic and purely non-atomic von Neumann algebras of type I (and any of those summands may vanish) we see that the proof of the first assertion from part (a) of Theorem 0.1 follows from the combination of Propositions 2.2, 2.4 with Proposition 1.6 and Corollary 1.7.

We shall now concentrate on part (b) of Theorem 0.1. It follows from Remark 2.3 that the first four spaces from (**L**) are pairwise non-isomorphic. Combining [AL], Theorem 6 with [M] (see also [GL]) we infer that none of the last five spaces from (**L**) is isomorphic to any of the first four spaces from (**L**). Listing for convenience the last five spaces from (**L**) as

$$(\mathbf{L}') \qquad S_p \oplus L_p,\ L_p(S_p),\ C_p \oplus L_p,\ L_p(C_p),\ C_p \oplus L_p(S_p),$$

we note that the first two spaces from (**L**$'$) are L_p-spaces associated with finite von Neumann algebras, whereas the last three spaces are L_p-spaces associated with non-finite von Neumann algebras. By [Su2], Corollary 3.3, it follows that none of the last three spaces from ($\mathbf{L}'$) is isomorphic to any of the first two spaces from ($\mathbf{L}'$). Thus, to complete the proof of part (b) of Theorem 0.1 we need to show only that each of the following three couples

$$(S_p \oplus L_p,\ L_p(S_p)), \quad (C_p \oplus L_p,\ L_p(C_p)), \quad (C_p \oplus L_p,\ C_p \oplus L_p(S_p))$$

consists of non-isomorphic spaces. The latter fact will follow immediately from the following theorem.

THEOREM 2.5: *For any $p \in [1,2)$, the Banach space $L_p(S_p)$ cannot be isomorphically embedded into $C_p \oplus L_p$.*

Proof: Suppose, contrapositively, that there exists an isomorphism T from $L_p(S_p)$ into $C_p \oplus L_p$. Fix some positive integer n and some real $r \in (p,2)$. We shall denote by π_n the natural isometrical embedding of C_p^n into S_p. Let e_{ij}, $1 \le i,j \le n$, be the element of C_p^n whose matrix has only one non-zero entry, namely 1 in the (i,j)-th place. Let P_1 (respectively, P_2) be the canonical projection from $C_p \oplus L_p$ onto C_p (respectively, L_p). Let

$$(f_n)_{n=1}^{\infty} \subseteq L_p$$

be a normalized basic sequence in L_p which is 1-equivalent to the unit vector basis $(g_n)_{n=1}^{\infty}$ of l_r (see, for example, [LT 2] Corollary 2.f.5). Given a pair of indices (i,j), $1 \le i,j \le n$, the sequence

$$(f_k \otimes \pi_n(e_{ij}))_{k=1}^{\infty} \subseteq L_p(S_p)$$

is a normalized basic sequence in $L_p(S_p)$ which is still 1-equivalent to $(g_n)_{n=1}^{\infty}$ and we either have

$$\|P_1 T(f_k \otimes \pi_n(e_{ij}))\|_{C_p} \to 0, \tag{2.1}$$

or

$$\|P_1 T(f_k \otimes \pi_n(e_{ij}))\|_{C_p} \not\to 0. \tag{2.2}$$

If (2.1) holds, then passing to a subsequence we may further achieve that

$$\|P_1 T(2^{-m/r} \sum_{k=2^m+1}^{2^{m+1}} f_k \otimes \pi_n(e_{ij}))\|_{C_p} \to 0. \tag{2.3}$$

If (2.2) holds, then passing to a subsequence and relabelling if necessary, we see that the sequence

$$(P_1 T(f_{k_m^{(i,j)}} \otimes \pi_n(e_{ij})))_{m=1}^{\infty}$$

is a basic sequence in C_p which is either equivalent to the unit vector basis of l_p, or that of l_2, $(e_k)_{k=1}^{\infty}$ (see [AL] Theorem 1). It is easy to see that $(P_1 T(f_k \otimes \pi_n(e_{ij})))_{k=1}^{\infty}$ is not equivalent to the unit vector basis of l_p. Indeed, if it were the case, then for some constant $C > 0$ and any positive natural N we have

$$\begin{aligned} CN^{1/p} &\leq \|\sum_{k=1}^{N} P_1 T(f_k \otimes \pi_n(e_{ij}))\|_{C_p} \\ &\leq \|P_1 T\| \cdot \|\sum_{k=1}^{N} f_k \otimes \pi_n(e_{ij})\|_{L_p(S_p)} \\ &= \|P_1 T\| \cdot \|\sum_{k=1}^{N} g_k\|_{l_r} \\ &= \|P_1 T\| \cdot N^{1/r} \end{aligned}$$

which for sufficiently large N contradicts the assumption $r \in (p, 2)$. Thus, if (2.2) holds, then (again passing to a subsequence and relabelling if necessary) we may assume that there exists a constant $K \geq 1$ such that $(P_1 T(f_k \otimes \pi_n(e_{ij})))_{k=1}^{\infty}$ is K-equivalent to $(e_k)_{k=1}^{\infty}$, in particular

$$\begin{aligned} \|P_1 T(2^{-m/r} \sum_{k=2^m+1}^{2^{m+1}} f_k \otimes \pi_n(e_{ij}))\|_{C_p} &\leq K \cdot 2^{-m/r} \cdot \|\sum_{k=2^m+1}^{2^{m+1}} e_i\|_{l_2} \\ &= K \cdot 2^{m/2 - m/r} \to 0, \end{aligned}$$

in other words (2.3) still holds. Obviously, an arbitrary subsequence of $(f_n)_{n=1}^{\infty}$ is again 1-equivalent to $(g_n)_{n=1}^{\infty}$ and the latter fact enables us to repeat the arguments given above consecutively for all n^2 pairs of the indices (i, j) (each time passing to a subsequence if necessary). Thus, we may assume that (2.3) holds for every pair (i, j), $1 \leq i, j \leq n$.

We set

$$u_m := 2^{-m/r} \sum_{k=2^m+1}^{2^{m+1}} f_k.$$

It it is clear that

$$\|u_m\|_{L_p} = 2^{-m/r} \|\sum_{k=2^m+1}^{2^{m+1}} g_k\|_{l_r} = 2^{-m/r} \cdot 2^{m/r} = 1$$

and, further, that the space

$$[u_k \otimes \pi_n(e_{ij})]_{i,j=1}^n$$

is isometrically isomorphic to C_p^n for each $k = 1, 2, \ldots$. At the same time it follows from (2.3) that

$$\|P_1 T(u_k \otimes \pi_n(e_{ij}))\|_{C_p} \to 0, \quad \text{as } k \to \infty$$

for every pair (i,j), $1 \le i,j \le n$. It further follows that for given $\epsilon > 0$, $n \in \mathbb{N}$ there exists an integer k_0 such that

$$\|P_1 T(u_k \otimes \pi_n(x))\|_{C_p} < \epsilon \|T\|$$

for every $k \ge k_0$ and every $x \in C_p^n$, $\|x\|_{C_p} \le 1$. It immediately follows that for sufficiently large k the operator

$$T_n := P_2 T|_{[u_k \otimes \pi_n(e_{ij})]_{1\le i,j\le n}} : [u_k \otimes \pi_n(e_{ij})]_{1\le i,j\le n} \longrightarrow L_p$$

satisfies

$$\|T_n x\|_{L_p} \ge (1-\epsilon)\|Tx\|_{C_p \oplus L_p}, \quad \forall x \in [u_k \otimes \pi_n(e_{ij})]_{1\le i,j\le n}.$$

In other words, the operator T_n is invertible, and the norm of its inverse does not exceed $(1-\epsilon)^{-1}\|T^{-1}\|$. Taking into account that

$$\|T_n\| = \|P_2 T|_{[u_k \otimes \pi_n(e_{ij})]_{1\le i,j\le n}}\| \le \|T\|$$

we arrive at the fact that T_n is an isomorphic embedding of C_p^n $(= [u_k \otimes \pi_n(e_{ij})]_{1\le i,j\le n})$ into L_p such that

$$\sup_n \|T_n\| \cdot \|T_k^{-1}\| < \infty.$$

This contradicts [GL] (see also [P] Theorem 2.1, Remark 2.3) and it completes the proof of Theorem 2.5. ∎

3. Part (c) of Theorem 0.1

To complete the proof of part (c) of Theorem 0.1 we need only show that the spaces

$$L_1(C_1) \quad \text{and} \quad C_1 \oplus L_1(S_1)$$

are non-isomorphic. A tool to be employed for doing so is the Dunford–Pettis property.

Recall that a Banach space E is said to have **the Dunford–Pettis property** if every weakly compact operator defined on E sends weakly compact sets into norm compact subsets, or equivalently, whenever (x_n) and (f_n) are weakly null sequences in E and E^* respectively, we have

$$\lim_n f_n(x_n) = 0.$$

This definition is due to Grothendieck [G] and goes back to a classical result of Dunford and Pettis [DP] which says that all $L_1(\Omega, \tau)$-spaces have this property. Chu and Iochum proved in [CI] that if $\mathcal{M}$ is a finite von Neumann algebra of type I, then the predual of $\mathcal{M}$ has the Dunford–Pettis property. The latter fact will be used in the proof of the following theorem.

THEOREM 3.1: *There is no complemented isomorphic copy of the Banach space $L_1(l_2)$ in $C_1 \oplus L_1(S_1)$.*

Since C_1 contains complemented isomorphic copies of l_2, we immediately derive from Theorem 3.1 the following corollary which asserts somewhat more than just the non-isomorphism of $L_1(C_1)$ and $C_1 \oplus L_1(S_1)$.

COROLLARY 3.2: *There is no complemented isomorphic copy of the Banach space $L_1(C_1)$ in $C_1 \oplus L_1(S_1)$.*

Proof of Theorem 3.1: Suppose, contrapositively, that there exists a linear isomorphism

$$T\colon L_1(l_2) \to X$$

where X is a complemented subspace of $C_1 \oplus L_1(S_1)$, i.e. there exists a closed subspace Y of $C_1 \oplus L_1(S_1)$ such that

$$X \oplus Y \approx C_1 \oplus L_1(S_1).$$

Recall that $C_1 \oplus L_1(S_1)$ is the predual of the von Neumann algebra

$$B(l_2) \oplus (L_\infty \otimes \mathcal{N}) \ (= B(l_2) \oplus (L_\infty \otimes (\bigoplus_{n=1}^{\infty} B(l_2^n))))$$

and also (see [DU]) that $(L_1(l_2))^* = L_\infty(l_2)$. Let P_1 (respectively, P_2) be the canonical projection from $C_1 \oplus L_1(S_1)$ onto C_1 (respectively, $L_1(S_1)$). Let Q_1 (respectively, Q_2) be the canonical projection from $B(l_2) \oplus (L_\infty \otimes \mathcal{N})$ onto $B(l_2)$ (respectively, $L_\infty \otimes \mathcal{N}$). For any $x \in C_1 \oplus L_1(S_1)$ and $y \in B(l_2) \oplus (L_\infty \otimes \mathcal{N})$, we have

$$\langle x, y\rangle = \langle P_1 x, Q_1 y\rangle + \langle P_2 x, Q_2 y\rangle. \tag{3.1}$$

Assume for the moment that we have constructed two sequences

$$(u_j)_{j=1}^\infty \subseteq L_1(l_2) \quad \text{and} \quad (v_j)_{j=1}^\infty \subseteq L_\infty(l_2)$$

satisfying the following four conditions:

(A) $$\|u_j\|_{L_1(l_2)} \le 1, \quad \|v_j\|_{L_\infty(l_2)} \le 1,$$

(B) $$\sigma(L_1(l_2), L_\infty(l_2)) - \lim_j u_j = 0, \quad \sigma(L_\infty(l_2), L_\infty(l_2)^*) - \lim_j v_j = 0,$$

(C) $$\langle u_j, v_j\rangle \not\to 0,$$

and

(D) $$\|P_1 T(u_j)\|_{C_1} \to 0.$$

Then a contradiction may be obtained as follows. First of all note that (B) implies that

$$\sigma(X^*, X^{**}) - \lim_j (T^*)^{-1}(v_j) = 0$$

and since

$$X^* \oplus Y^* \approx (C_1 \oplus L_1(S_1))^*, \quad X^{**} \oplus Y^{**} \approx (C_1 \oplus L_1(S_1))^{**}$$

we have also

$$\sigma\big((C_1 \oplus L_1(S_1))^*, (C_1 \oplus L_1(S_1))^{**}\big) - \lim_j (T^*)^{-1}(v_j) = 0,$$

or

$$\sigma\big(B(l_2) \oplus (\mathcal{N} \otimes L_\infty), (B(l_2) \oplus (\mathcal{N} \otimes L_\infty))^*\big) - \lim_j (T^*)^{-1}(v_j) = 0.$$

We get immediately that

$$\sigma\big(B(l_2) \oplus (\mathcal{N} \otimes L_\infty), (B(l_2) \oplus (\mathcal{N} \otimes L_\infty))^*\big) - \lim_j Q_2(T^*)^{-1}(v_j) = 0. \tag{3.2}$$

Next note that (B) implies

$$\sigma(L_1(S_1), L_1(S_1)^*) - \lim_j P_2T(u_j) = 0. \tag{3.3}$$

Thus, it follows from (3.1), (3.2), (3.3), (A), (B), (D) and from the Dunford–Pettis property of the space $L_1(S_1)$ that

$$\begin{aligned}\langle T(u_j), (T^*)^{-1}(v_j)\rangle &= \langle P_1T(u_j) + P_2T(u_j), Q_1(T^*)^{-1}(v_j) + Q_2(T^*)^{-1}(v_j)\rangle \\ &= \langle P_1T(u_j), Q_1(T^*)^{-1}(v_j)\rangle + \langle P_2T(u_j), Q_2(T^*)^{-1}(v_j)\rangle \\ &\to 0. \end{aligned}\tag{3.4}$$

But (3.4) contradicts (C), since

$$\langle T(u_j), (T^*)^{-1}(v_j)\rangle = \langle u_j, v_j\rangle.$$

We shall construct the sequences $(u_j)_{j=1}^{\infty}$ and $(v_j)_{j=1}^{\infty}$ using the sequence

$$(f_j \otimes e_j)_{j=1}^{\infty} \subseteq L_1(l_2)$$

where, as before, $(e_n)_{n=1}^{\infty}$ is the unit vector basis of l_2 and $(f_n)_{n=1}^{\infty}$ is a normalized basic sequence in L_1 which is 1-equivalent to the unit vector basis $(g_n)_{n=1}^{\infty}$ of l_r, $r \in (1,2)$.

LEMMA 3.3: *The sequence $(f_j \otimes e_j)_{j=1}^{\infty}$ forms a Schauder basis of the Banach space $[f_j \otimes e_j]_{j=1}^{\infty} \subseteq L_1(l_2)$ which is equivalent to $(g_n)_{n=1}^{\infty}$.*

Proof of Lemma 3.3: It follows from [LT 2] 1.d.6 that there exists a positive constant C such that for an arbitrary positive integer n and an arbitrary sequence of scalars $(\alpha_j)_{j=1}^{n}$ we have

$$\begin{aligned}\Big\|\sum_{j=1}^{n} \alpha_j f_j \otimes e_j\Big\|_{L_1(l_2)} &= \Big\|\Big(\sum_{j=1}^{n} |\alpha_j f_j|^2\Big)^{1/2}\Big\|_{L_1} \\ &\le C\Big\|\sum_{j=1}^{n} \alpha_j f_j\Big\|_{L_1} \\ &= C\Big\|\sum_{j=1}^{n} \alpha_j g_j\Big\|_{l_r} \\ &\le C^2\Big\|\Big(\sum_{j=1}^{n} |\alpha_j f_j|^2\Big)^{1/2}\Big\|_{L_1} \\ &= C^2\Big\|\sum_{j=1}^{n} \alpha_j f_j \otimes e_j\Big\|_{L_1(l_2)}. \end{aligned}$$ ∎

We first show how the required sequences $(u_j)_{j=1}^{\infty}$ and $(v_j)_{j=1}^{\infty}$ may be defined in the case when

$$\|P_1 T(f_j \otimes e_j)\|_{C_1} \to 0. \tag{3.5}$$

If (3.5) is indeed the case, then we simply set

$$u_j := f_j \otimes e_j, \quad v_j := \operatorname{sgn}(f_j) \otimes e_j.$$

Indeed, the condition (D) clearly holds and, since

$$\|u_j\|_{L_1(l_2)} = \|v_j\|_{L_\infty(l_2)} = \langle u_j, v_j \rangle = 1,$$

the conditions (A) and (C) are satisfied as well. The fact that

$$\sigma(L_1(l_2), L_\infty(l_2)) - \lim_j f_j \otimes e_j = 0$$

follows immediately from Lemma 3.3 and thus the first condition in (B) is satisfied. To see that

$$\sigma(L_\infty(l_2), L_\infty(l_2)^*) - \lim_j \operatorname{sgn}(f_j) \otimes e_j = 0$$

we note that the linear map

$$\beta\colon l_2 \to L_\infty(l_2)$$

given by

$$\beta(\sum_{j=1}^{\infty} \alpha_j e_j) := \sum_{j=1}^{\infty} \alpha_j \operatorname{sgn}(f_j) \otimes e_j$$

is bounded. Since $\sigma(l_2, l_2^*) - \lim_j e_j = 0$ we have also

$$\sigma(L_\infty(l_2), L_\infty(l_2)^*) - \lim_j v_j = \sigma(L_\infty(l_2), L_\infty(l_2)^*) - \lim_j \beta(e_j) = 0$$

and the second condition in (B) is also satisfied.

To complete the proof of Theorem 3.1, we have to construct sequences $(u_j)_{j=1}^{\infty}$ and $(v_j)_{j=1}^{\infty}$ assuming that

$$\|P_1 T(f_j \otimes e_j)\|_{C_1} \nrightarrow 0. \tag{3.6}$$

If indeed (3.6) is the case, then (passing to a subsequence if necessary) we may assume that $(P_1 T(f_j \otimes e_j))_{j=1}^{\infty}$ is a basic sequence in C_1 (see, for example, [LT 1]).

Hence, by [AL] Theorem 1, the sequence $(P_1T(f_j \otimes e_j))_{j=1}^{\infty}$ contains a subsequence which is equivalent either to the unit vector basis in l_1, or to the unit vector basis in l_2. By Lemma 3.3, there exists a constant $C \geq 1$ such that for any positive integer n and any sequence $(k_j)_{j=1}^{n}$ we have

$$\begin{aligned}\|\sum_{j=1}^{n} P_1T(f_{k_j} \otimes e_{k_j})\|_{C_1} &\leq \|P_1T\| \cdot \|\sum_{j=1}^{n} f_{k_j} \otimes e_{k_j}\|_{L_1(l_2)} \\ &\leq C\|P_1T\| \cdot \|\sum_{j=1}^{n} g_{k_j}\|_{l_r} = C\|P_1T\| \cdot n^{1/r}\end{aligned}$$

and this implies that the basic sequence $(P_1T(f_j \otimes e_j))_{j=1}^{\infty}$ does not contain any subsequence which is equivalent to the unit vector basis in l_1. Therefore (again passing to a subsequence if necessary) we may assume that there exists a constant $K \geq 1$ such that $(P_1T(f_j \otimes e_j))_{j=1}^{\infty}$ is K-equivalent to $(e_j)_{j=1}^{\infty}$. In particular

$$\|P_1T(\sum_{i=2^j+1}^{2^{j+1}} f_i \otimes e_i)\|_{C_1} \leq K\|\sum_{i=2^j+1}^{2^{j+1}} e_i\|_{l_2} = K \cdot 2^{j/2}. \tag{3.7}$$

In this situation we set

$$u_j := (CK)^{-1} \cdot 2^{-j/r} \sum_{i=2^j+1}^{2^{j+1}} f_i \otimes e_i.$$

We have (see the proof of Lemma 3.3)

$$\|u_j\|_{L_1(l_2)} \leq K^{-1} \cdot 2^{-j/r} \|\sum_{i=2^j+1}^{2^{j+1}} g_i\|_{l_r} = K^{-1} \cdot 2^{-j/r} \cdot 2^{j/r} \leq K^{-1} \tag{3.8}$$

and

$$\|u_j\|_{L_1(l_2)} \geq C^{-1}(CK)^{-1} \cdot 2^{-j/r} \|\sum_{i=2^j+1}^{2^{j+1}} g_i\|_{l_r} = K^{-1}C^{-2}. \tag{3.9}$$

For any $j \geq 1$, the element u_j belongs to the space $L_1([e_i]_{i=2^j+1}^{2^{j+1}}) \subseteq L_1(l_2)$ and it follows from the usual Hahn–Banach Theorem that there exists an element

$$v_j = \sum_{i=2^j+1}^{2^{j+1}} \beta_i \otimes e_i \in L_\infty([e_i]_{i=2^j+1}^{2^{j+1}}), \quad (\beta_i)_{i=2^j+1}^{2^{j+1}} \subseteq L_\infty \tag{3.10}$$

such that

$$\|v_j\|_{L_\infty(l_2)} \le 1, \tag{3.11}$$

and (see the estimate in (3.9))

$$\langle u_j, v_j \rangle \ge \frac{1}{2} K^{-1} C^{-2}. \tag{3.12}$$

It follows from (3.8), (3.11) and (3.12) that for the sequences $(u_j)_{j=1}^{\infty}$ and $(v_j)_{j=1}^{\infty}$ conditions (A) and (C) are satisfied. We have also (D) since (see (3.7))

$$\|P_1 T(u_j)\|_{C_1} \le C^{-1} \cdot 2^{-j/r} \| \sum_{i=2^j+1}^{2^{j+1}} e_i \|_{l_2} = C^{-1} \cdot 2^{-j/r} \cdot 2^{j/2} \to 0, \quad \text{as } j \to \infty.$$

A moment of reflection shows that the first convergence in (B) follows from Lemma 3.3. The second convergence in (B) may be established along the same lines as earlier. Indeed, we may define

$$\beta' \colon l_2 \to L_\infty(l_2)$$

by

$$\beta'\Big(\sum_{j=1}^{\infty} \alpha_j e_j\Big) := \sum_{j=1}^{\infty} \alpha_j v_j.$$

It follows from (3.10), (3.11) that β' is bounded and it further implies the second convergence in (B). This completes the proof of Theorem 3.1. ∎

Remark 3.4: We conjecture that the answer to the question whether there are isomorphic embeddings of the space $L_1(C_1)$ into $C_1 \oplus L_1(S_1)$ is negative.

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